CHAPTER 9

C2: Radian measure

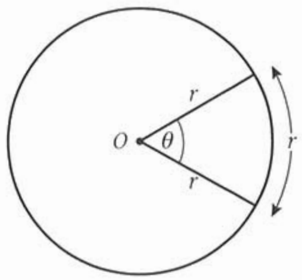
Learning objectives

After studying this chapter, you should be able to:

- understand what is meant by a radian
- convert between degrees and radians
- recall and use the formula for the length of an arc of a circle
- recall and use the formula for the area of a sector of a circle.

9.1 Radians as a unit of measure of angles

You will have measured angles in degrees, where 1 degree is $\frac{1}{360}$ of a complete turn. In higher level mathematics a degree is not the most appropriate unit to use to measure angles. It is more useful to measure an angle in units called radians.

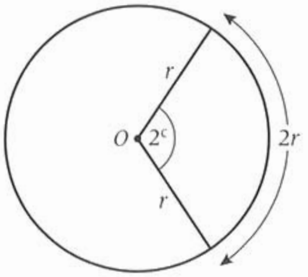


One radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle. In the diagram on the right, $\theta = 1$ radian.

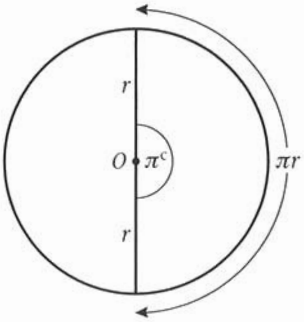
1 radian is written as 1 rad or as 1^c	1 radian $\approx 57°$

9.2 Changing between degrees and radians

An arc of length $2r$ subtends an angle of 2 radians at the centre of the circle.



An arc of length πr subtends an angle of π radians at the centre of the circle.



An arc of length $2\pi r$ subtends an angle of 2π radians at the centre of the circle. The circumference of the circle is $2\pi r$ which subtends an angle of 360° at the centre of the circle.
So $360° = 2\pi$ rads.

Worked example 9.1

Convert 20° to radians.

Since 20° is $\frac{1}{18}$ of 360° we could have written $\frac{1}{18}$ of 2π rads $= \frac{\pi}{9}$.

Solution

$360° = 2\pi$ rads.

$$1° = \frac{2\pi}{360} \text{ rads.}$$

$$20° = 20 \times \frac{2\pi}{360} \text{ rads} = \frac{\pi}{9} \text{ rads } \{= 0.34906 \ldots \text{rads}\}$$

If an exact answer is needed, or an answer in terms of π, give $\frac{\pi}{9}$ rads as the final answer.

Note: The expression $\frac{\pi}{9}$ is usually read as 'pi by nine' – no doubt as an abbreviation for 'pi divided by nine'.

Worked example 9.2

Convert $\frac{3\pi}{5}$ radians to degrees.

Solution

$$2\pi \text{ rads} = 360°$$

$$1 \text{ rad} = \frac{360°}{2\pi} \{= 57.2957 \ldots°\}$$

$$\frac{3\pi}{5} \text{ rads} = \frac{3\pi}{5} \times \frac{360°}{2\pi} = 108°$$

EXERCISE 9A

1 Change the following to degrees:

(a) $\frac{\pi}{2}$ rads, **(b)** 4π rads, **(c)** $\frac{\pi}{3}$ rads, **(d)** $\frac{\pi}{4}$ rads,

(e) $\frac{5\pi}{6}$ rads, **(f)** $\frac{3\pi}{4}$ rads, **(g)** $\frac{2\pi}{3}$ rads, **(h)** $\frac{11\pi}{6}$ rads,

(i) $\frac{7\pi}{4}$ rads, **(j)** $\frac{5\pi}{2}$ rads.

2 Convert the following to radians. Give your answers in terms of π.

(a) 180°, **(b)** 120°, **(c)** 36°, **(d)** 24°,
(e) 108°, **(f)** 540°, **(g)** 80°, **(h)** 225°,
(i) 405°, **(j)** 15°.

3 Convert the following to degrees, giving your answers to 3 sf.

(a) 2 rads, **(b)** 0.5 rads, **(c)** 1.8 rads, **(d)** 3 rads,
(e) 0.3 rads, **(f)** 2.3 rads, **(g)** 1.28 rads, **(h)** 1.6 rads,
(i) $\frac{7\pi}{11}$ rads, **(j)** $\frac{3\pi}{7}$ rads.

4 Convert the following to radians, giving your answers to 3 sf.

(a) 60°, **(b)** 150°, **(c)** 25°, **(d)** 305°,
(e) 96°, **(f)** 78°, **(g)** 14°, **(h)** 82°,
(i) 38°, **(j)** 500°.

5 Use your calculator to find the value, to 3 sf, of
(i) $\sin\theta$, **(ii)** $\cos\theta$, **(iii)** $\tan\theta$
for the following values of θ in radians:

(a) 0.4, **(b)** 1.2, **(c)** 2, **(d)** $\frac{5\pi}{8}$.

Remember to set your calculator to radian mode.

6 Find the angle in radians turned in exactly 1 day by:

(a) the hour hand, **(b)** the minute hand

of a clock. Leave your answers in terms of π.

7 Find the obtuse angle, in radians, between the hands of a clock showing

(a) 4 p.m, **(b)** 2:30 a.m, **(c)** 3:45 p.m.

Leave your answers in terms of π.

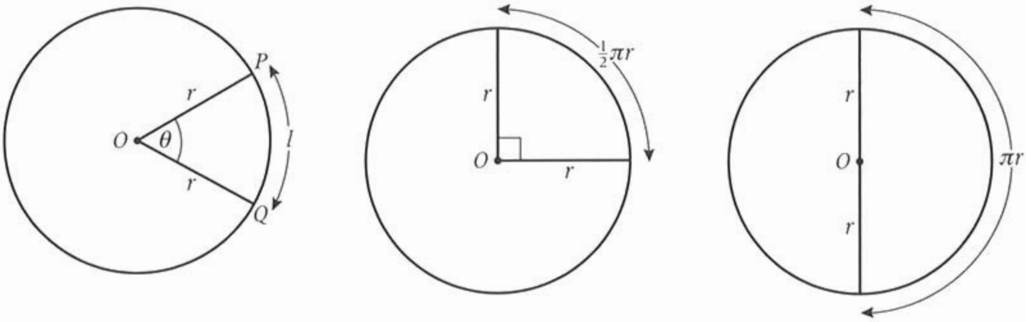
8 By considering a right-angled triangle, show that:

(a) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$,

(b) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

9.3 Arc length of a circle

Let arc PQ of a circle subtend an angle θ at the centre of the circle. Suppose the length of the arc is l then l is directly proportional to θ. (As θ doubles the arc length doubles.)



$$l \propto \theta \Rightarrow \quad l = k\theta$$

When $\theta = 2\pi$, $l = 2\pi r \Rightarrow 2\pi r = k2\pi$

$$\Rightarrow \quad k = r$$

$$\Rightarrow \quad l = r\theta, \text{ where } \theta \text{ is in radians}$$

> The length l of an arc of a circle is given by $l = r\theta$, where r is the radius and θ is the angle, in radians, subtended by the arc at the centre of the circle.

Worked example 9.3

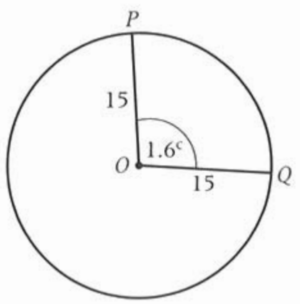
An arc PQ of a circle of radius 15 cm subtends an angle of 1.6 radians at the centre, O, of the circle.

(a) Find the perimeter of the sector OPQ.

(b) Find the length of the chord PQ.

A sector of a circle is the region bounded by two radii and an arc. The larger region is called the major sector, the smaller region is the minor sector.

Solution



(a) Using $l = r\theta$,

length of arc $PQ = 15 \times 1.6 = 24$ cm.

Perimeter of sector $OPQ = 15 + 15 + 24$

$= 54$ cm

Forgetting to include the lengths of the two radii is a common error.

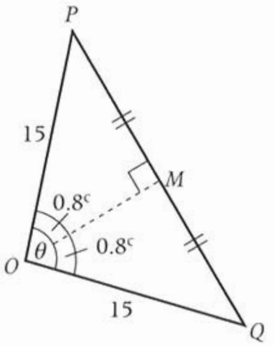
(b) OPQ is an isosceles triangle.

Chord $PQ = 2 \times PM = 2 \times r\sin(\theta/2)$

$= 2 \times 15 \sin 0.8^c$

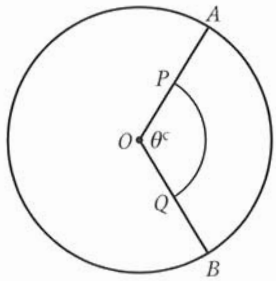
$= 30 \times 0.717\,35 \ldots$

$= 21.52 \ldots = 21.5$ cm (to 3 sf)



Worked example 9.4

OA and *OB* are radii of a circle. Arc *AB* subtends an angle of θ radians at *O*. *P* and *Q* are the mid-points of *OA* and *OB*, respectively. *PQ* is an arc of a circle with centre *O*. Given that the perimeter of sector *OAB* = perimeter of *ABQP*, show that $\theta = 2$.



Solution

Let $OA = OB = 2x$

Perimeter of sector $OAB = OA + \text{arc } AB + BO$

$= 2x + 2x\theta + 2x$

Perimeter of $ABQP = \text{arc } AB + QB + \text{arc } QP + PA$

$= 2x\theta + x + x\theta + x$

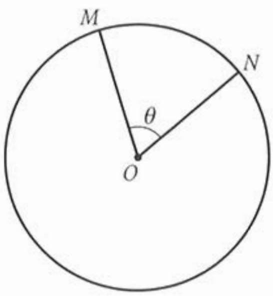
Perimeter of sector OAB = perimeter of $ABQP$

$\Rightarrow 4x + 2x\theta = 2x + 3x\theta$

$\Rightarrow 2x = x\theta \Rightarrow \theta = 2.$

EXERCISE 9B

[In this exercise, *M* and *N* are points on the circumference of a circle with centre *O* and θ is the angle subtended by the minor arc, *MN*, at *O*.]



1 Find the length of the minor arc *MN* in each of the following cases:

(a) $OM = 18$ cm, $\theta = 1.5$ rads,

(b) $ON = 20$ cm, $\theta = 2.5$ rads,

(c) $ON = 7$ cm, $\theta = 0.3$ rads,

(d) $OM = 5.5$ cm, $\theta = 1.4$ rads.

2 Find the perimeter of the **major** sector OMN for each part of Question 1.

3 Find the angle subtended by the minor arc MN in each of the following:

(a) arc $MN = 21$ cm, $OM = 10$ cm,

(b) arc $MN = 30$ cm, $ON = 12$ cm,

(c) arc $MN = 10$ cm, $ON = 16$ cm,

(d) arc $MN = 16$ cm, $OM = 6$ cm,

(e) $OM = 10$ cm, perimeter of minor sector $OMN = 30$ cm,

(f) $OM = 4$ cm, perimeter of minor sector $OMN = 10$ cm,

(g) $OM = 12$ cm, perimeter of minor sector $OMN = 50$ cm,

(h) $OM = 5$ cm, perimeter of minor sector $OMN = 18$ cm.

4 Find the radius of a circle whose minor arc, MN, subtends an angle θ at the centre of the circle in each of the following:

(a) arc $MN = 10$ cm, $\theta = 2$ rads,

(b) arc $MN = 8$ cm, $\theta = 2.5$ rads,

(c) arc $MN = 16$ cm, $\theta = 10$ rads,

(d) arc $MN = 18$ cm, $\theta = 30°$,

(e) arc $MN = 20$ cm, $\theta = 80°$,

(f) arc $MN = 8\pi$ cm, $\theta = \frac{\pi}{4}$ rads.

5 The perimeter of the minor sector OMN is the same as the length of the major arc MN. Show that the minor arc MN subtends an angle $(\pi - 1)$ radians at O.

6 Given that $OM = 6$ m and $\theta = \frac{\pi}{3}$ radians, show that the perimeter of the minor sector OMN is about 283 mm longer than the perimeter of triangle OMN.

7 Find the length of the chord, MN, to the nearest millimetre, in each of the following cases:

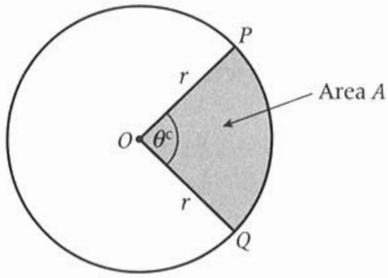
(a) arc $MN = 20$ cm, $OM = 10$ cm,

(b) arc $MN = 8$ cm, $ON = 8$ cm,

(c) arc $MN = 18$ cm, $OM = 12$ cm,

(d) arc $MN = 45$ cm, $ON = 15$ cm.

9.4 Area of a sector of a circle



Arc PQ, which subtends an angle θ at the centre, O, of the circle and radii OP and OQ form a sector of the circle. Suppose the area of the sector is A, then A is directly proportional to θ.

$$A \propto \theta \Rightarrow A = k\,\theta$$

When $\theta = 2\pi$, $A = \pi r^2 \Rightarrow \pi r^2 = k2\pi$

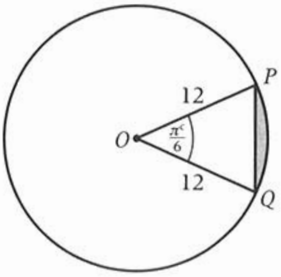
$$\Rightarrow k = \frac{1}{2}r^2$$

$$\Rightarrow A = \frac{1}{2}r^2\,\theta\text{, where }\theta\text{ is in radians}$$

> The area A of a sector of a circle is given by $\frac{1}{2}r^2\,\theta$, where r is the radius and θ is the angle, in radians, subtended by the arc at the centre of the circle.

Worked example 9.5

Find the area of the shaded segment.



A segment of a circle is the region bounded by an arc and its chord.

Solution

Area of shaded segment = area of sector POQ − area of triangle POQ

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

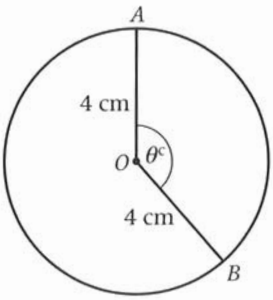
$$= \frac{1}{2}(12)^2\frac{\pi}{6} - \frac{1}{2}(12)^2\sin\frac{\pi}{6}$$

$$= 12\,\pi - 36 = 1.70\text{ cm}^2\text{ (to 3 sf)}$$

Use $\frac{1}{2}ab\sin C$ for area of triangle.

Worked example 9.6

The area of the minor sector OAB is 24 cm^2. The radius of the circle is 4 cm. Find the length of the **major** arc AB.



Solution

Using $\quad A = \frac{1}{2}r^2\theta,$

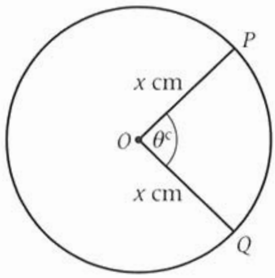
$$24 = \frac{1}{2}(4)^2\theta \Rightarrow \theta = 3$$

Length of minor arc $AB = r\theta = 4 \times 3 = 12$.

Length of **major** arc $AB = 2\pi r - 12 = 8\pi - 12 = 13.1$ cm (to 3 sf).

Worked examination question 9.7

A circular sector, of area A cm^2, has bounding radii, each of length x cm, and the angle between these radii is θ radians. Given that the perimeter of the sector is 12 cm,



(a) express θ in terms of x,

(b) show that $A = 6x - x^2$. [A]

Solution

(a) Perimeter of sector $= OP + \text{arc } PQ + OQ$

$$12 = x + x\theta + x$$

$$\Rightarrow \quad 12 - 2x = x\theta$$

$$\Rightarrow \quad \theta = \frac{12 - 2x}{x}$$

(b) $A = \frac{1}{2}r^2\theta = \frac{1}{2}x^2\theta = \frac{1}{2}x(x\theta) = \frac{1}{2}x(12 - 2x) = 6x - x^2$.

EXERCISE 9C

[In Questions **1–4**, E and F are points on the circumference of a circle with centre O and θ is the angle subtended at O by the minor arc EF.]

1 Find the area of the minor sector, OEF, in each of the following:

(a) $OE = 4$ cm, $\theta = 2$ rads,

(b) $OF = 6$ cm, $\theta = 1.5$ rads,

(c) $OE = 1.5$ cm, $\theta = 0.75$ rads,

(d) $OF = 1$ cm, $\theta = 60°$.

2 The area of the minor sector, OEF, is A cm^2. Find the length of OE in each of the following:

(a) $A = 16$, $\theta = 2$ rads,

(b) $A = 25$, $\theta = 0.5$ rads,

(c) $A = 2.25$, $\theta = 0.125$ rads,

(d) $A = 5$, $\theta = 30°$.

3 Find the area of the **major** segment, OEF, in each of the following:

(a) $OE = 10$ cm, $\theta = \frac{\pi}{6}$ rads,

(b) $OE = 6$ cm, $\theta = \frac{\pi}{3}$ rads,

(c) $OE = 2.4$ cm, $\theta = 2$ rads,

(d) $OF = 4.5$ cm, $\theta = 72°$.

4 Find the length of the major arc EF in each of the following:

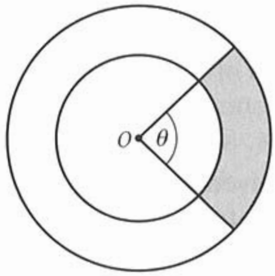
(a) $OF = 8$ cm, area of minor sector $OEF = 40\text{ cm}^2$,

(b) $OE = 6$ cm, area of minor sector $OEF = 18\text{ cm}^2$,

(c) $OF = 10$ cm, area of minor sector $OEF = 42\text{ cm}^2$,

(d) $OE = 12$ cm, area of minor sector $OEF = 30\text{ cm}^2$.

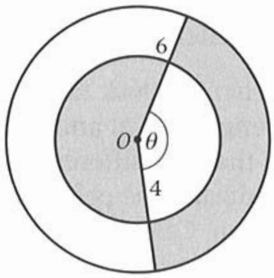
5 The diagram shows two concentric circles with centre O and radii 4 cm and 6 cm. Find the area of the shaded region in each of the following:



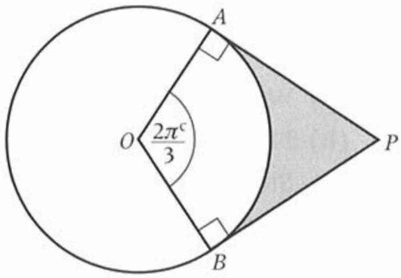
(a) $\theta = \dfrac{\pi}{2}$ rads, **(b)** $\theta = \dfrac{\pi}{3}$ rads,

(c) $\theta = 2$ rads, **(d)** $\theta = 120°$.

6 The diagram shows two concentric circles with centre O and radii 4 cm and 6 cm. The areas of the two shaded regions are equal. Show that $\theta = \dfrac{8\pi}{9}$ radians.



7 Points A and B lie on the circumference of a circle of radius 9 cm and centre O such that angle $AOB = \dfrac{2\pi}{3}$ radians. PA and PB are tangents to the circle. Find the area of the region bounded by the arc AB and the tangents PA and PB.



8 The perimeter of a sector of a circle of radius r and angle θ is the same as the perimeter of a square of side r.

(a) Show that $\theta = 2$ radians.

(b) Show that the area of the sector is the same as the area of the square. [A]

MIXED EXERCISE

1 The area of a sector of a circle of radius 10 cm is 75 cm^2. Find the arc length of this sector. [A]

2 A circle with centre O and radius 8 cm passes through points A and B such that OAB is an equilateral triangle. The point P is on the major arc AB. Find, in terms of π,

(a) the size of angle APB,

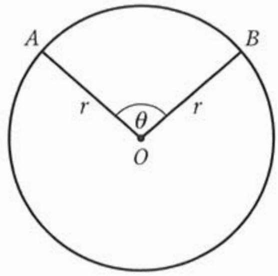
(b) the perimeter of the major segment ABP.

3 The perimeter of a sector OAB of a circle of centre O and radius 25 cm is 80 cm.

(a) Calculate the area of the sector OAB.

(b) AB cuts the circle into two segments. Calculate the area of the **major** segment.

4 The diagram shows a circular path with centre O and radius r, together with two other paths along the radii AO and OB. The size of the angle AOB is θ radians, where $\theta < \pi$. The widths of the paths may be neglected in the calculations. Peter runs along the radii AO and OB, then along the minor arc BA. Mary runs along the major arc AB.



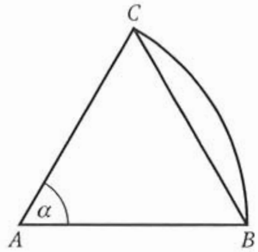
(a) Given that Peter and Mary run the same distance, show that

$$\theta = \pi - 1.$$

(b) Given that they each run 410 metres, find the radius of the circular path correct to the nearest metre. [A]

5 On a circular clock face, with centre O, the minute hand OA is of length 10 cm and the hour hand OB is of length 6 cm. Prove that the difference between the distances that A and B travel during the period of 1 hour from 12 o'clock to 1 o'clock is 19π cm. Calculate, correct to the nearest millimetre, how far A is from B at 1 o'clock. [A]

6 The diagram shows an equilateral triangle ABC with sides of length 6 cm and an arc BC of a circle with centre A.



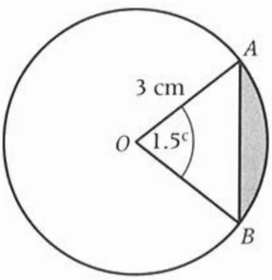
(a) Write down, in radians, the value of the angle α.

(b) Find the length of the arc BC.

(c) Show that the area of the triangle ABC is $9\sqrt{3}$ cm^2.

(d) Show that the area of the sector ABC is 6π cm^2. [A]

7 The diagram shows a circle with centre O and radius 3 cm. The points A and B on the circle are such that the angle AOB is 1.5 radians.

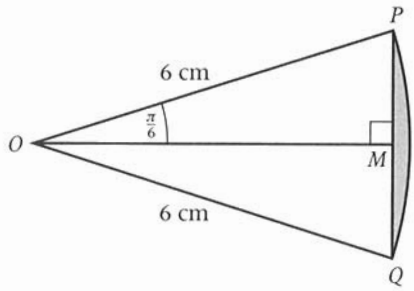


(a) Find the length of the minor arc AB.

(b) Find the area of the minor sector OAB.

(c) Show that the area of the shaded segment is approximately 2.3 cm^2. [A]

8 The diagram shows a sector of a circle, with centre O and radius 6 cm. The mid-point of the chord PQ is M, and the angle $POM = \dfrac{\pi}{6}$ radians.



(a) Write down the exact values of:

(i) the lengths of PM and OM,

(ii) the length of the arc PQ,

(iii) the area of the sector POQ.

(b) Use the appropriate answers from **(a)** to show that the area of the shaded region is $m(2\pi - 3\sqrt{3})$ cm^2, for some integer m whose value is to be determined. [A]

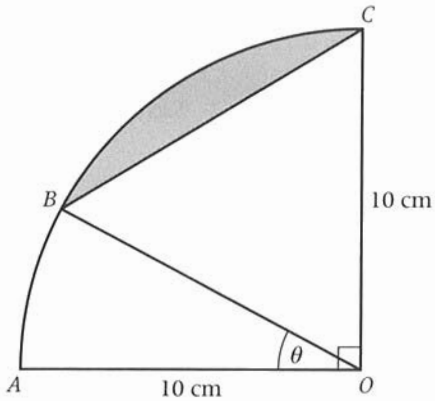
9 A circle PQR has centre O and radius r. It is divided into three equal areas by the chords PQ and PR. Given that angle $QPR = \theta$ radians, find the area of

(a) the sector OQR,

(b) the triangle OPQ.

Deduce that $\sin\theta = \dfrac{\pi}{3} - \theta$. [A]

10 The diagram shows an arc ABC of a circle centre O and radius 10 cm.



Angle $AOB = \theta$ radians and angle AOC is a right angle.

The shaded segment is bounded by the arc BC and the chord BC.

(a) Write down, in terms of θ, the area of the sector AOB.

(b) Show that the area of the shaded segment is $25(\pi - 2\theta - 2\cos\theta)$ cm^2.

(c) Given that the length of the arc BC is 4 times the length of the arc AB, show that $\theta = \dfrac{\pi}{10}$. [A]

Key point summary

1 One radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle. *p337*

1 radian is written as 1 rad or as 1^c

1 rad $\approx 57°$

2 Converting between degrees and radians use $360° = 2\pi$ rads. *p337*

Hence $x° = x \times \frac{2\pi}{360}$ rads.

3 The length l of an arc of a circle is given by $l = r\,\theta$, where r is the radius and θ is the angle, in radians, subtended by the arc at the centre of the circle. *p340*

4 The area A of a sector of a circle is given by $A = \frac{1}{2}r^2\theta$, where r is the radius and θ is the angle, in radians, subtended by the arc at the centre of the circle. *p343*

Test yourself | What to review

[*In this section give final numerical answers to 3 sf*]

Test yourself	What to review
1 In triangle DEF, angle $D = 1.8$ radians, angle $E = \frac{\pi}{4}$ radians and angle $F = \theta$ radians. Find the value of $\cos\theta$.	*Section 9.2*
2 Points A and B lie on the circumference of a circle of radius 8 cm and centre O. Angle AOB, subtended by the minor arc AB, is 1.4 radians. **(a)** Find the area of the minor sector AOB. **(b)** Find the area of the minor segment. **(c)** Find the perimeter of the **major** sector.	*Section 9.3 and 9.4*
3 The perimeter of a sector of a circle of radius r cm is P cm. The area of the sector is A cm^2. Show that $A = \frac{1}{2}r(P - 2r)$.	*Sections 9.3 and 9.4*

Test yourself ANSWERS

1 0.849.

2 **(a)** 44.8 cm^2; **(b)** 13.3 cm^2; **(c)** 55.1 cm.